

Branching form of the resolvent at threshold for discrete Laplacians

Kenichi ITO (Kobe University)
joint work with
Arne JENSEN (Aalborg University)

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Introduction: Discrete Laplacian

- **Thresholds generated by critical values**

For any function $u: \mathbb{Z}^d \rightarrow \mathbb{C}$ define $\Delta u: \mathbb{Z}^d \rightarrow \mathbb{C}$ by

$$(\Delta u)[n] = \sum_{j=1}^d \left(u[n + e_j] + u[n - e_j] - 2u[n] \right) \quad \text{for } n \in \mathbb{Z}^d.$$

The operator $H_0 = -\Delta$ is bounded and self-adjoint on $\mathcal{H} = \ell^2(\mathbb{Z}^d)$. It has spectrum

$$\sigma(H_0) = \sigma_{\text{ac}}(H_0) = [0, 4d],$$

and **thresholds**

$$\tau(H_0) = \{0, 4, \dots, 4d\}.$$

Let $\widehat{\mathcal{H}} = L^2(\mathbb{T}^d)$, $\mathbb{T} = \mathbb{R}/(2\pi\mathbb{Z})$, and define the Fourier transform $\mathcal{F}: \mathcal{H} \rightarrow \widehat{\mathcal{H}}$ and its inverse $\mathcal{F}^*: \widehat{\mathcal{H}} \rightarrow \mathcal{H}$ by

$$(\mathcal{F}u)(\theta) = (2\pi)^{-d/2} \sum_{n \in \mathbb{Z}^d} e^{-in\theta} u[n],$$

$$(\mathcal{F}^*f)[n] = (2\pi)^{-d/2} \int_{\mathbb{T}^d} e^{in\theta} f(\theta) d\theta,$$

and then

$$\mathcal{F}H_0\mathcal{F}^* = \Theta(\theta) = 2d - 2\cos\theta_1 - \dots - 2\cos\theta_d.$$

Since $\partial_j \Theta(\theta) = 2\sin\theta_j$, the critical points of signature (p, q) are

$$\gamma(p, q) = \left\{ \theta \in \{0, \pi\}^d; \# \{ \theta_j = 0 \} = p, \# \{ \theta_j = \pi \} = q \right\}.$$

Hence the critical values $0, 4d$ are thresholds of **elliptic type**, and $4, \dots, 4(d-1)$ are those of **hyperbolic type**.

○ **Purpose: Asymptotic expansion of resolvent**

Q. Can we compute an asymptotic expansion of the resolvent

$$R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as } z \rightarrow 4q \text{ for } q = 0, 1, \dots, d?$$

Note that the resolvent $R_0(z)$ has a **convolution kernel**:

$$R_0(z)u = k(z, \cdot) * u; \quad k(z, n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z} d\theta.$$

A. Yes. By localizing around $\gamma(p, q) \subset \mathbb{T}^d$ and changing variables the situation reduces to that for an **ultra-hyperbolic operator**.

- As far as we know, an explicit asymptotics of $R_0(z)$ around a threshold seems to have been **open** except for 0 and $4d$.

Ultra-hyperbolic operator (a model operator)

Consider an ultra-hyperbolic operator on \mathbb{R}^d :

$$\square = \partial_1^2 + \cdots + \partial_p^2 - \partial_{p+1}^2 - \cdots - \partial_{p+q}^2; \quad p, q \geq 0, \quad d = p + q.$$

The operator $H_0 = -\square$ is self-adjoint on $\mathcal{H} = L^2(\mathbb{R}^d)$ with

$$\mathcal{D}(H_0) = \{u \in \mathcal{H}; \square u \in \mathcal{H} \text{ in the distributional sense}\}.$$

It has spectrum

$$\sigma(H_0) = \sigma_{\text{ac}}(H_0) = \begin{cases} [0, \infty) & \text{if } (p, q) = (d, 0), \\ (-\infty, 0] & \text{if } (p, q) = (0, d), \\ \mathbb{R} & \text{otherwise,} \end{cases}$$

and a single threshold

$$\tau(H_0) = \{0\}.$$

Using the Fourier transform $\mathcal{F}: \mathcal{H} \rightarrow \mathcal{H}$ and its inverse $\mathcal{F}^*: \mathcal{H} \rightarrow \mathcal{H}$, we can write

$$\mathcal{F}H_0\mathcal{F}^* = \Xi(\xi) = \xi'^2 - \xi''^2; \quad \xi = (\xi', \xi'') \in \mathbb{R}^p \oplus \mathbb{R}^q.$$

The only critical point is $\xi = 0$, and the associated critical value, or a threshold 0 is said to be

1. of elliptic type if $(p, q) = (d, 0)$ or $(0, d)$;
2. of hyperbolic type otherwise.

Q'. Can we compute an asymptotic expansion of the resolvent

$$R_0(z) = (H_0 - z)^{-1} \sim ?? \quad \text{as } z \rightarrow 0?$$

A'. **Yes.** In particular, **square root**, **logarithm** and **dilogarithm branchings** show up, depending on **parity of (p, q)** .

◦ Square root, logarithm and dilogarithm

We always choose branches of \sqrt{w} and $\log w$ such that

$$\begin{aligned} \operatorname{Im} \sqrt{w} &> 0 & \text{for } w \in \mathbb{C} \setminus [0, \infty), \\ -\pi < \operatorname{Im} \log w < \pi & & \text{for } w \in \mathbb{C} \setminus (-\infty, 0], \end{aligned}$$

respectively. In addition, let us set for $w \in \mathbb{C} \setminus [1, \infty)$

$$\operatorname{Li}_1(w) = -\log(1 - w), \quad \operatorname{Li}_2(w) = \int_0^w \frac{\operatorname{Li}_1(\lambda)}{\lambda} d\lambda,$$

which have the Taylor expansions: For $|w| < 1$

$$\operatorname{Li}_1(w) = \sum_{k=1}^{\infty} \frac{w^k}{k}, \quad \operatorname{Li}_2(w) = \sum_{k=1}^{\infty} \frac{w^k}{k^2}.$$

○ Elliptic operator in odd dimensional space

Theorem. Let d be *odd*, $(p, q) = (d, 0)$, and $\gamma > 0$. Then

$$k_\gamma(z, x) = \frac{i\pi}{2}(\sqrt{z})^{d-2}e(\sqrt{z}x) - \frac{1}{2} \int_{\tilde{\Gamma}(\gamma)} \frac{\rho^{d-1}e(\rho x)}{\rho^2 - z} d\rho,$$

where $\tilde{\Gamma}(r) = \{re^{i\theta} \in \mathbb{C}; \theta \in [0, \pi]\}$.

○ Elliptic operator in even dimensional space

Theorem. Let d be *even*, $(p, q) = (d, 0)$, and $\gamma > 0$. Then

$$k_\gamma(z, x) = -\frac{1}{2}(\sqrt{z})^{d-2}e(\sqrt{z}x) \operatorname{Li}_1\left(\frac{\gamma^2}{z}\right) + \frac{1}{2} \int_0^{\gamma^2} \frac{(\sqrt{\lambda})^{d-2}e(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}e(\sqrt{z}x)}{\lambda - z} d\lambda.$$

Proposition. 1. The function $e(\zeta)$ is *even and entire* in $\zeta \in \mathbb{C}^d$,
and

$$e(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} e_\alpha \zeta^{2\alpha}; \quad e_\alpha = \frac{2}{2^d \pi^{d/2}} \frac{(-1/4)^{|\alpha|}}{\alpha! \Gamma(|\alpha| + d/2)}.$$

2. For any $z \in \mathbb{C}$ the function $e(\sqrt{z}x)$ satisfies the *eigenequation*

$$(-\Delta - z)e(\sqrt{z}x) = 0; \quad \Delta = \square_{d,0}.$$

Here a branch of \sqrt{z} does not matter, since $e(\zeta)$ is even.

○ **Hyperbolic case with odd-even or even-odd signature**

Theorem. Let (p, q) be *odd-even or even-odd*, and $\gamma > 0$. Then

$$k_\gamma(z, x) = \frac{i\pi}{2}(\sqrt{z})^{d-2}\psi_+(\sqrt{z}x) + \chi_\gamma(z, x),$$

where

$$\begin{aligned} \chi_\gamma(z, x) = & -\frac{1}{2} \int_{\Gamma(\gamma)} \frac{\tau^{d-1} \psi_+(\tau x)}{\tau^2 - z} d\tau + \frac{1}{2} \int_{\Gamma(\gamma)} \frac{\tau^{d-1} \psi_-(\tau x)}{\tau^2 + z} d\tau \\ & + \frac{1}{4} \int_{i^2 \tilde{\Gamma}(\gamma^2)} \frac{h_{+, \gamma}(\lambda, x)}{\lambda - z} d\lambda \end{aligned}$$

with

$$h_{\pm, \gamma}(\tau^2, x) = \tau^{d-2} \int_{-\gamma}^{\gamma} \frac{f_{\pm}(\sigma/\tau, \tau x)}{\sigma} d\sigma.$$

○ **Hyperbolic case with even-even signature**

Theorem. Let (p, q) be *even-even*, and $\gamma > 0$. Then

$$k_\gamma(z, x) = -\frac{1}{2}(\sqrt{z})^{d-2}\psi_+(\sqrt{z}x) \operatorname{Li}_1\left(\frac{\gamma^4}{z^2}\right) + \chi_\gamma(z, x),$$

where

$$\begin{aligned} \chi_\gamma(z, x) = & \frac{1}{2} \left(\int_0^{\gamma^2} - \int_{-\gamma^2}^0 \right) \frac{(\sqrt{\tau})^{d-2}\psi_+(\sqrt{\tau}x) - (\sqrt{z})^{d-2}\psi_+(\sqrt{z}x)}{\tau - z} d\tau \\ & + \frac{1}{2} \int_{-\gamma^2}^{\gamma^2} \frac{h_{+, \gamma}(\lambda, x)}{\lambda - z} d\lambda \end{aligned}$$

with

$$h_{\pm, \gamma}(\tau^2, x) = \tau^{d-2} \int_\tau^\gamma \frac{f_\pm(\sigma/\tau, \tau x)}{\sigma} d\sigma - \tau^{d-2}\psi_\pm(\tau x).$$

○ Hyperbolic case with odd-odd signature

Theorem. Let (p, q) be *odd-odd*, and $\gamma > 0$. Then

$$k_\gamma(z, x) = -\frac{1}{4}(\sqrt{z})^{d-2}\phi_+(\sqrt{z}x) \left[\text{Li}_2\left(\frac{\gamma^2}{z}\right) - \text{Li}_2\left(-\frac{\gamma^2}{z}\right) \right] + \chi_\gamma(z, x),$$

where

$$\begin{aligned} \chi_\gamma(z, x) = & \frac{1}{4} \int_0^{\gamma^2} \frac{(\sqrt{\lambda})^{d-2}\phi_+(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}\phi_+(\sqrt{z}x)}{\lambda - z} \log\left(\frac{\gamma^2}{\lambda}\right) d\lambda \\ & + \frac{1}{4} \int_{-\gamma^2}^0 \frac{(\sqrt{\lambda})^{d-2}\phi_+(\sqrt{\lambda}x) - (\sqrt{z})^{d-2}\phi_+(\sqrt{z}x)}{\lambda - z} \log\left(-\frac{\gamma^2}{\lambda}\right) d\lambda \\ & + \frac{1}{2} \int_{-\gamma^2}^{\gamma^2} \frac{h_{+, \gamma}(\lambda, x)}{\lambda - z} d\lambda \end{aligned}$$

$$\text{with } h_{\pm, \gamma}(\tau^2, x) = \tau^{d-2} \int_\tau^\gamma \frac{f_\pm(\sigma/\tau, \tau x) - \phi_\pm(\tau x)}{\sigma} d\sigma.$$

○ **Properties of $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$**

The functions $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$ are **entire in $\zeta \in \mathbb{C}^d$** , and

$$\phi_{\pm}(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left(\sum_{a \in J_{\alpha}} f_{\pm, \alpha, a} \right),$$

$$\psi_{\pm}(\zeta) = \sum_{\alpha \in \mathbb{Z}_+^d} \zeta^{2\alpha} \left(\sum_{a \in I_{\alpha} \setminus J_{\alpha}} \frac{[i^{2|\alpha| - 2|a| + d - 2} - 1] f_{\pm, \alpha, a}}{2|\alpha| - 2|a| + d - 2} \right),$$

where

$$I_{\alpha} = \left\{ a \in \mathbb{Z}_+^2; 0 \leq a \leq (2|\alpha'| + p - 1, 2|\alpha''| + q - 1) \right\},$$

$$J_{\alpha} = \left\{ a \in I_{\alpha}; |a| = |\alpha| + (d - 2)/2 \right\},$$

$$f_{\pm, \alpha, a} = \frac{(\pm 1)^{a'} (\mp 1)^{a''} e'_{\alpha'} e''_{\alpha''}}{2^{2|\alpha| + d - 2}} \binom{2|\alpha'| + p - 1}{a'} \binom{2|\alpha''| + q - 1}{a''}.$$

○ Properties of $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$, continued

The functions $\phi_{\pm}(\zeta)$ and $\psi_{\pm}(\zeta)$ satisfy

1. $\phi_{\pm}(\zeta) = 0$ and $\psi_{\pm}(\zeta) = \psi_{\pm}(-\zeta)$ if (p, q) is odd-even or even-odd;
2. $\phi_{\pm}(\zeta) = 0$ and $\psi_{\pm}(\zeta) = -i^{d-2}\psi_{\mp}(i\zeta)$ if (p, q) is even-even;
3. $\phi_{\pm}(\zeta) = i^{d-2}\phi_{\mp}(i\zeta)$ and $\psi_{\pm}(\zeta) = 0$ if (p, q) is odd-odd.

In addition, for any $z \in \mathbb{C}$

$$(-\square \mp z)\phi_{\pm}(\sqrt{z}x) = 0, \quad (-\square \mp z)\psi_{\pm}(\sqrt{z}x) = 0.$$

○ Outline of the results for ultra-hyperbolic operator

Theorem. 1. If (p, q) is *odd-even or even-odd*, there exist operators $F(z), G(z)$ analytic at $z = 0$ such that

$$R_0(z) = F(z)\sqrt{z} + G(z).$$

2. If (p, q) is *even-even*, there exist operators $F(z), G(z)$ analytic at $z = 0$ such that

$$R_0(z) = F(z)\operatorname{Li}_1\left(\frac{1}{z}\right) + G(z).$$

3. If (p, q) is *odd-odd*, there exist operators $F(z), G(z)$ analytic at $z = 0$ such that

$$R_0(z) = F(z)\left[\operatorname{Li}_2\left(\frac{1}{z}\right) - \operatorname{Li}_2\left(-\frac{1}{z}\right)\right] + G(z).$$

- **“Very rough” strategy for proof**

The resolvent has a limiting **convolution** expression

$$(R_0(z)u)(x) = \lim_{\gamma \rightarrow \infty} \int_{\mathbb{R}^d} k_\gamma(z, x - y) u(y) dy \quad \text{for } u \in \mathcal{S}(\mathbb{R}^d);$$

$$k_\gamma(z, x) = (2\pi)^{-d} \int_{|\xi'| + |\xi''| < \gamma} \frac{e^{ix\xi}}{\xi'^2 - \xi''^2 - z} d\xi.$$

It suffices to expand the kernel $k_\gamma(z, x)$, since it contains all the singular part of $R_0(z)$.

If we move on to the spherical or hyperbolic coordinates, a singular part of $k_\gamma(z, x)$ takes, more or less, the **standard form**

$$I = \int_0^\gamma \frac{a(\rho)}{\rho^2 - z} d\rho.$$

There could appear only the following three types of $a(\rho)$:

- If $a(\rho) = 2b(\rho^2)$ with b analytic, then

$$I = \int_{-\gamma}^{\gamma} \frac{b(\rho^2)}{(\rho - \sqrt{z})(\rho + \sqrt{z})} d\rho = i\pi \frac{b(z)}{\sqrt{z}} - \int_{|z|=\gamma, \operatorname{Im} z \geq 0} \frac{b(\rho^2)}{\rho^2 - z} d\rho.$$

- If $a(\rho) = 2\rho b(\rho^2)$ with b analytic, then

$$I = \int_0^{\gamma} \frac{b(\lambda)}{\lambda - z} d\lambda = \int_0^{\gamma} \frac{b(z)}{\lambda - z} d\lambda + \int_0^{\gamma} \frac{b(\lambda) - b(z)}{\lambda - z} d\lambda$$

- If $a(\rho) = 2\rho b(\rho^2)(\log \rho^2)$ with b analytic, then

$$I = \int_0^{\gamma} \frac{b(\lambda) \log \lambda}{\lambda - z} d\lambda = \int_0^{\gamma} \frac{b(z) \log \lambda}{\lambda - z} d\lambda + \int_0^{\gamma} \frac{[b(\lambda) - b(z)] \log \lambda}{\lambda - z} d\lambda$$

Precise results for discrete Laplacian

- Localization around critical points

Note that $R_0(z)$ has a **convolution kernel**:

$$k(z, n) = (2\pi)^{-d} \int_{\mathbb{T}^d} \frac{e^{in\theta}}{\Theta(\theta) - z} d\theta.$$

Denote the set of all the **critical points** of signature (p, q) by

$$\gamma(p, q) = \{\theta^{(l)}\}_{l=1, \dots, L}; \quad L = \#\gamma(p, q) = \binom{d}{p} = \binom{d}{q},$$

Take **neighborhoods** $U_l \subset \mathbb{T}^d$ of $\theta^{(l)}$, and then decompose

$$k(z, n) = k_0(z, n) + k_1(z, n) + \dots + k_L(z, n);$$

$$k_l(z, n) = (2\pi)^{-d} \int_{U_l} \frac{e^{in\theta}}{\Theta(\theta) - z} d\theta \quad \text{for } l = 1, \dots, L.$$

We let, e.g.,

$$\theta^{(1)} = (0, \dots, 0, \pi, \dots, \pi) \in \mathbb{T}^p \times \mathbb{T}^q = \mathbb{T}^d.$$

Take local coordinates $\xi(\theta) = \xi^{(1)}(\theta)$ around $\theta^{(1)} \in \mathbb{T}^d$:

$$\xi_j(\theta) = \begin{cases} 2 \sin(\theta_j/2) & \text{for } j = 1, \dots, p, \\ 2 \cos(\theta_j/2) & \text{for } j = p+1, \dots, p+q, \end{cases}$$

and set

$$\mathcal{U}_1 = \{\theta \in \mathbb{T}^d; |\xi'(\theta)| + |\xi''(\theta)| < 2\}.$$

Then we can write

$$k_1(z, n) = (2\pi)^{-d} \int_{|\xi'| + |\xi''| < 2} \frac{e^{in\theta(\xi)}}{\xi'^2 - \xi''^2 - (z - 4q)} \frac{d\xi}{\prod_{j=1}^d (1 - \xi_j^2/4)^{1/2}}.$$

We will do the same construction for $k_2(z, n), \dots, k_L(z, n)$.

Theorem. Let $(p, q) = (d, 0)$.

1. If d is *odd*, there exists a function $\chi(z, n)$ analytic in $z \in \Delta(4)$ such that

$$k(z, n) = \frac{i\pi}{2}(\sqrt{z})^{d-2}e(\sqrt{z}, n) + \chi(z, n).$$

2. If d is *even*, there exists a function $\chi(z, n)$ analytic in $z \in \Delta(4)$ such that

$$k(z, n) = -\frac{1}{2}(\sqrt{z})^{d-2}e(\sqrt{z}, n) \operatorname{Li}_1\left(\frac{1}{z}\right) + \chi(z, n).$$

Proposition. 1. The function $e(\rho, \mathbf{n})$ is *analytic* in $\rho \in \Delta(2)$, and

$$e(\rho, \mathbf{n}) = \frac{2}{2^d \pi^{d/2}} \sum_{k=0}^{\infty} \rho^{2k} \sum_{\alpha \in \mathbb{Z}_+^d, |\alpha|=k} \frac{\prod_{j=1}^d (1/2 - n_j)_{\alpha_j} (1/2 + n_j)_{\alpha_j}}{4^{|\alpha|} \alpha! \Gamma(|\alpha| + d/2)},$$

where $(\nu)_k := \Gamma(\nu + k)/\Gamma(\nu)$ is the *Pochhammer symbol*. In particular, $e(z, \mathbf{n})$ can be expressed by the *Lauricella function*:

$$e(\rho, \mathbf{n}) = \frac{2}{2^d \pi^{d/2} \Gamma(d/2)} F_B^{(d)} \left(\frac{1}{2} - n_1, \dots, \frac{1}{2} - n_d, \frac{1}{2} + n_1, \dots, \frac{1}{2} + n_d; \frac{d}{2}; \frac{\rho^2}{4}, \dots, \frac{\rho^2}{4} \right).$$

2. For any $z \in \Delta(4)$, as a function in $\mathbf{n} \in \mathbb{Z}^d$,

$$(-\Delta - z)e(\sqrt{z}, \mathbf{n}) = 0.$$

Theorem. 1. If (p, q) is *odd-even or even-odd*, then there exists $\chi(\cdot, n) \in C^\omega(\Delta(4))$ such that

$$k(w + 4q, n) = \frac{i\pi}{2}(\sqrt{w})^{d-2} \sum_{l=1}^L \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

2. If (p, q) is *even-even*, there exists $\chi(\cdot, n) \in C^\omega(\Delta(4))$ such that

$$k(w + 4q, n) = -\frac{1}{2}(\sqrt{w})^{d-2} \text{Li}_1\left(\frac{16}{w^2}\right) \sum_{l=1}^L \psi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

3. If (p, q) is *odd-odd*, there exists $\chi(\cdot, n) \in C^\omega(\Delta(4))$ such that

$$k(w + 4q, n) = -\frac{1}{4}(\sqrt{w})^{d-2} \left[\text{Li}_2\left(\frac{4}{w}\right) - \text{Li}_2\left(-\frac{4}{w}\right) \right] \sum_{l=1}^L \phi_+^{(l)}(\sqrt{w}, n) + \chi(w, n).$$

o **Properties of $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$**

The functions $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$ are **analytic in $\tau \in \Delta(2)$** , and

$$\phi_{\pm}(\tau, n) = \sum_{\alpha \in \mathbb{Z}_+^2} \tau^{2|\alpha|} \left(\sum_{a \in J_{\alpha}} f_{\pm, \alpha, a}[n] \right),$$

$$\psi_{\pm}(\tau, n) = \sum_{\alpha \in \mathbb{Z}_+^2} \tau^{2|\alpha|} \left(\sum_{a \in I_{\alpha} \setminus J_{\alpha}} \frac{[i^{2|\alpha| - 2|a| + d - 2} - 1] f_{\pm, \alpha, a}[n]}{2|\alpha| - 2|a| + d - 2} \right),$$

where

$$I_{\alpha} = \left\{ a \in \mathbb{Z}_+^2; 0 \leq a \leq (2\alpha' + p - 1, 2\alpha'' + q - 1) \right\},$$

$$J_{\alpha} = \left\{ a \in I_{\alpha}; |a| = |\alpha| + (d - 2)/2 \right\}.$$

$$f_{\pm, \alpha, a}[n] = \frac{(\pm 1)^{a'} (\mp 1)^{a''}}{2^{2|\alpha| + d - 2}} \binom{2\alpha' + p - 1}{a'} \binom{2\alpha'' + q - 1}{a''} e'_{\alpha'}[n'] e''_{\alpha''}[n''].$$

○ Properties of $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$, continued

The functions $\phi_{\pm}(\tau, n)$ and $\psi_{\pm}(\tau, n)$ satisfy

1. $\phi_{\pm}(\tau, n) = 0$, $\psi_{\pm}(\tau, n) = \psi_{\pm}(-\tau, n)$ and $\psi_{\pm}(\tau, n) = \psi_{\pm}(\tau, -n)$ if (p, q) is odd-even or even-odd;
2. $\phi_{\pm}(\tau, n) = 0$, $\psi_{\pm}(\tau, n) = -i^{d-2}\psi_{\mp}(i\tau, n)$ and $\psi_{\pm}(\tau, n) = \psi_{\pm}(\tau, -n)$ if (p, q) is even-even;
3. $\phi_{\pm}(\tau, n) = i^{d-2}\phi_{\mp}(i\tau, n)$, $\phi_{\pm}(\tau, n) = \phi_{\pm}(\tau, -n)$ and $\psi_{\pm}(\tau, n) = 0$ if (p, q) is odd-odd.

In addition, for any $w \in \Delta(4)$, as functions in $n \in \mathbb{Z}^d$,

$$(-\Delta \mp w)\phi_{\pm}(\sqrt{w}, n) = 0, \quad (-\Delta \mp w)\psi_{\pm}(\sqrt{w}, n) = 0.$$